Stabilization of Capacitated Matching Games

Lucy Verberk

Eindhoven University of Technology

Joint work with Matthew Gerstbrein and Laura Sanità

Cooperative Matching Games

In *Cooperative Matching Games* (CMG) we are given a graph $G = (V, E)$ with edge weights $w \in \mathbb{R}^E_{\geq 0}$ and vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$. The vertices of the graph represent players. We call an allocation $y \in \mathbb{R}^V_{\geq 0}$ *stable* if no coalition of players $S \subseteq V$ has an incentive to deviate. This is described by the inequalities

$$\sum_{v \in S} y_v \geq \nu(G[S])$$

for all $S \subseteq V$, where $G[S]$ is the subgraph of G induced by S, and $\nu(G)$ is the value of a maximum-weight c-matching in G, formally defined as

$$\nu(G) := \max\{w^\top x : x(\delta(v)) \leq c_v \ \forall v \in V, 0 \leq x \leq 1, x \in \mathbb{Z}\}.$$

The *core* of a CMG consists of all stable allocations of total value $\nu(G)$.

Network Bargaining Games

In *Network Bargaining Games* (NBG) we are again given a triple (G, w, c). The vertices of the graph represent players, the edges represent potential deals between players and the edge weights represent the values of the deals. Each player v can enter in at most c_v deals. For each deal, the involved players have to decide how to split the value of their deal. Hence, an *outcome* is naturally associated with a c-matching M, and a vector $a \in \mathbb{R}^{2E}_{\geq 0}$ that satisfies $a_{uv} + a_{vu} = w_{uv}$ if $uv \in M$, and $a_{uv} = a_{vu} = 0$ otherwise. An outcome is *stable* if no pair of players has an incentive to break the current outcome to enter in a deal with each other.

Equivalence and Counter Example

The value of a maximum-weight fractional c-matching, called $\nu_f(G)$, is defined as $\nu(G)$ but without the integrality constraints. In unit-capacity graphs the following are equivalent: [7, 3]

(1) the core is nonempty for CMG on G,

(2) there exists a stable outcome for NBG on G,

(3) $\nu(G) = \nu_f(G)$ (we say G is *stable*).

This equivalence does not extend to capacitated graphs.

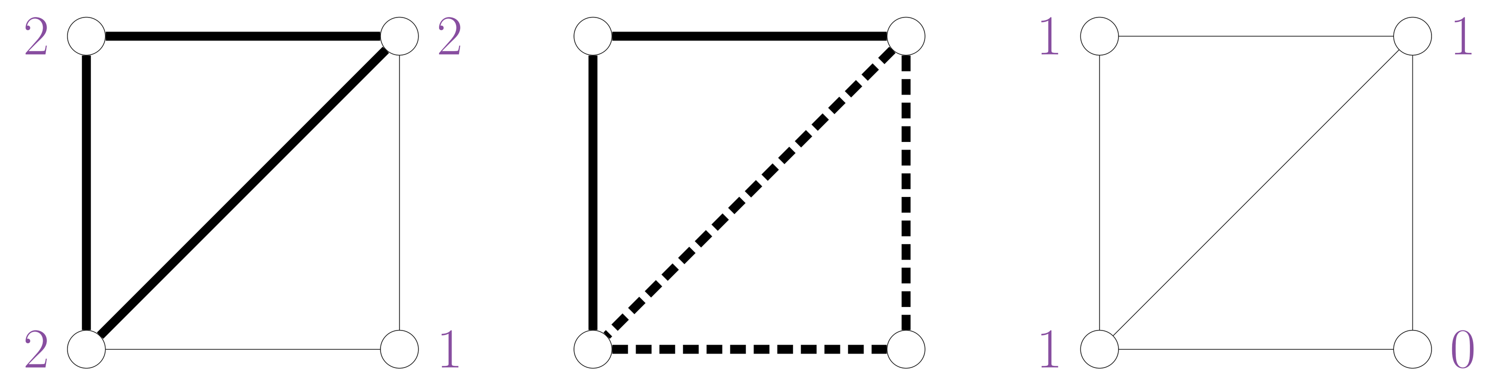
Theorem 1

For a given triple (G, w, c) the following holds:

$$(2) \iff (3) \implies (1),$$

but $(1) \not\Longrightarrow (3)$ and hence $(1) \not\Longrightarrow (2)$.

[2] gives $(2) \iff (3)$ and $(3) \implies (1)$. The following example shows $(1) \not\Longrightarrow (3)$.



$\nu(G) = 3$ $\nu_f(G) = 3.5$ $\in$ core

Open Problems

- Computational complexity of deciding when the core of a CMG is nonempty in capacitated graphs.
- Studying other ways to stabilize instances.
- Studying the stabilization problem for other games defined on graphs.

The Vertex-Stabilizer Problem

The vertex-stabilizer problem is motivated by the fact that not all graphs are stable. The goal is to minimally change a graph as to ensure a stable outcome. A natural way to change a graph is by removing vertices, i.e., blocking players.

Vertex-stabilizer problem: given a graph G, find a min-cardinality set $S \subseteq V$ s.t. $G \setminus S$ is stable.

A variation of the problem is the *M-vertex-stabilizer problem*, where the goal is to realize a given c-matching M as part of a stable outcome.

M-Vertex-stabilizer problem: given a graph G, find a min-cardinality set $S \subseteq V$ s.t. $G \setminus S$ is stable and M is a max-weight c-matching in $G \setminus S$.

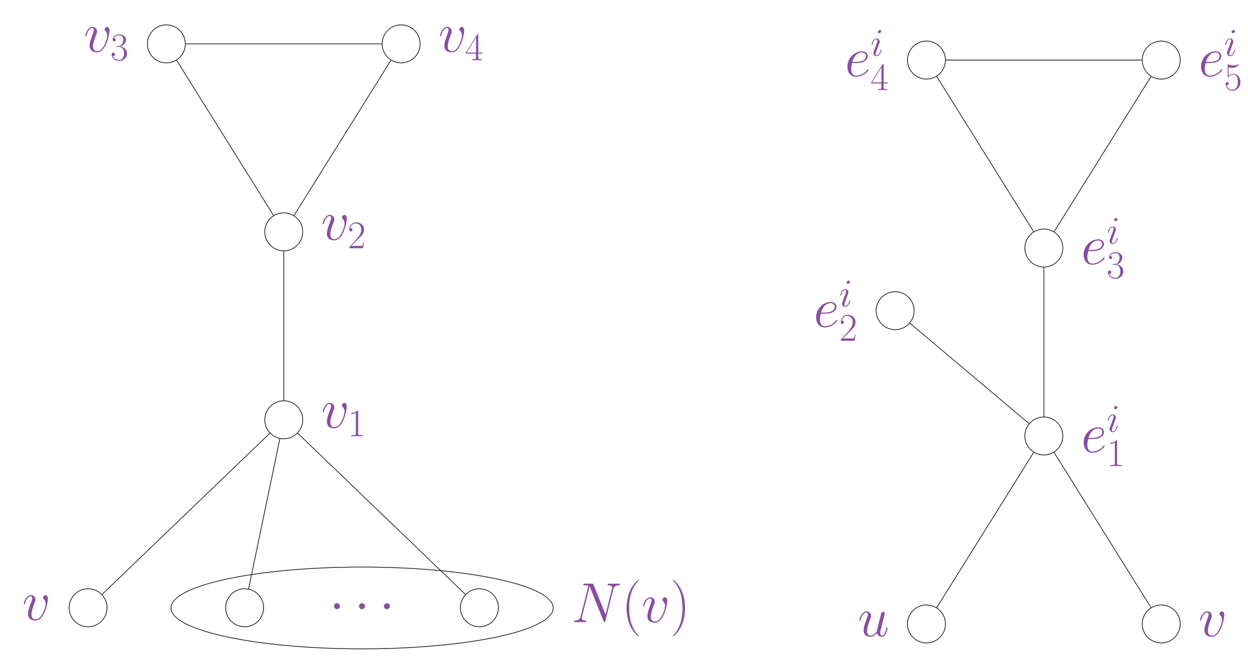
Both the vertex-stabilizer and M-vertex-stabilizer problem are polynomial-time solvable in unit-capacity graphs [1, 6, 8].

Vertex-Stabilizer in Capacitated Graphs

Theorem 2

Unless $P = NP$, the vertex-stabilizer problem cannot be approximated within a factor $n^{1-\varepsilon}$ for every $\varepsilon > 0$, even if all edges have unit-weight.

This inapproximability result is obtained by a reduction from the minimum independent dominating set problem [5].



Note that an n-approximation exists, e.g., by removing all vertices (but two).

M-Vertex-Stabilizer in Capacitated Graphs

Theorem 3

The M-vertex-stabilizer problem can be solved in polynomial-time.

This result is based on combining a reduction from capacitated graphs to unit-capacity graphs from [4], and a careful extension of the algorithmic results for the unit-capacity M-vertex-stabilizer problem from [8].

References

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