Stabilization of Capacitated Matching Games

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Cooperative Matching Games

In *Cooperative Matching Games* (CMG) we are given a graph $G = (V, E)$ with edge weights $w \in \mathbb{R}^E_{\geq 0}$ and vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$. The vertices of the graph represent players. We call an allocation $y \in \mathbb{R}^V_{\geq 0}$ ***stable*** if no coalition of players $S \subseteq V$ has an incentive to deviate. This is described by the inequalities

$$\sum_{v \in S} y_v \geq \nu(G[S])$$

for all $S \subseteq V$, where $G[S]$ is the subgraph of G induced by S, and $\nu(G)$ is the value of a maximum-weight c-matching in G, formally defined as

$$\nu(G) := \max\{w^\top x : x(\delta(v)) \leq c_v \ \forall v \in V, 0 \leq x \leq 1, x \in \mathbb{Z}\}.$$

The *core* of a CMG consists of all stable allocations of total value $\nu(G)$.

Network Bargaining Games

In *Network Bargaining Games* (NBG) we are again given a triple (G, w, c). The vertices of the graph represent players, the edges represent potential deals between players and the edge weights represent the values of the deals. Each player v can enter in at most c_v deals. For each deal, the involved players have to decide how to split the value of their deal. Hence, an *outcome* is naturally associated with a c-matching M, and a vector $a \in \mathbb{R}^{2E}_{\geq 0}$ that satisfies $a_{uv} + a_{vu} = w_{uv}$ if $uv \in M$, and $a_{uv} = a_{vu} = 0$ otherwise. An outcome is *stable* if no pair of players has an incentive to break the current outcome to enter in a deal with each other.

Equivalence and Counter Example

The value of a maximum-weight fractional c-matching, called $\nu_f(G)$, is defined as $\nu(G)$ but without the integrality constraints. In unit-capacity graphs the following are equivalent: [6, 2]

(1) the core is nonempty for CMG on G,

(2) there exists a stable outcome for NBG on G,

(3) $\nu(G) = \nu_f(G)$ (we say G is *stable*).

We show that this equivalence does not extend to capacitated graphs.

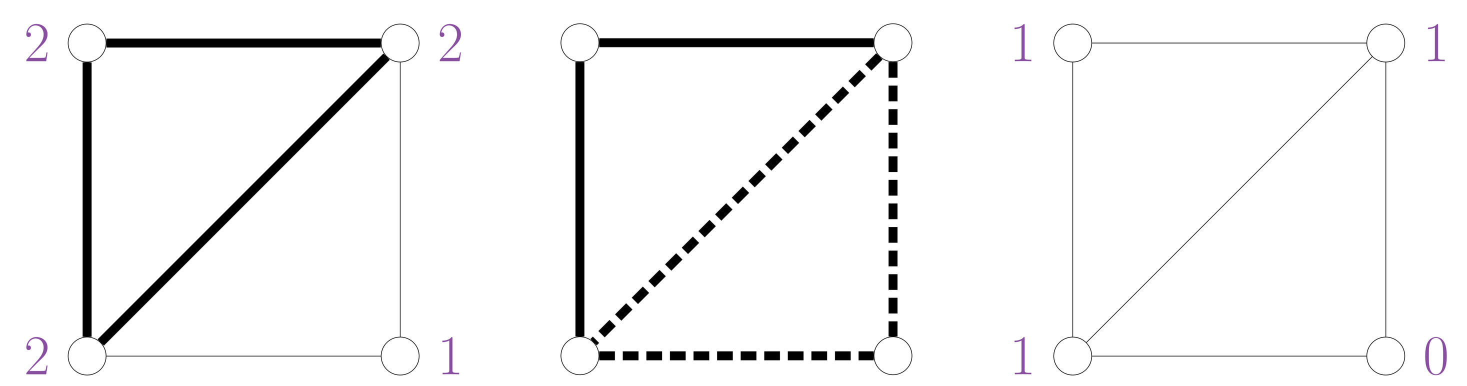
Theorem 1

For a given triple (G, w, c) the following holds:

$$(2) \iff (3) \implies (1),$$

but $(1) \not\Longrightarrow (3)$ and hence $(1) \not\Longrightarrow (2)$.

The following example shows $(1) \not\Longrightarrow (3)$.



The Vertex-Stabilizer Problem

The vertex-stabilizer problem is motivated by the fact that not all graphs are stable. The goal is to minimally change a graph as to guarantee the existence of a stable outcome for the corresponding NBG. A natural way to change a graph is by removing vertices, i.e., blocking players. A set of vertices, whose removal from the graph yields a stable graph, is called a vertex-stabilizer. The vertex-stabilizer problem asks to find a minimum-cardinality vertex-stabilizer.

Vertex-Stabilizer in Unit-Capacity Graphs

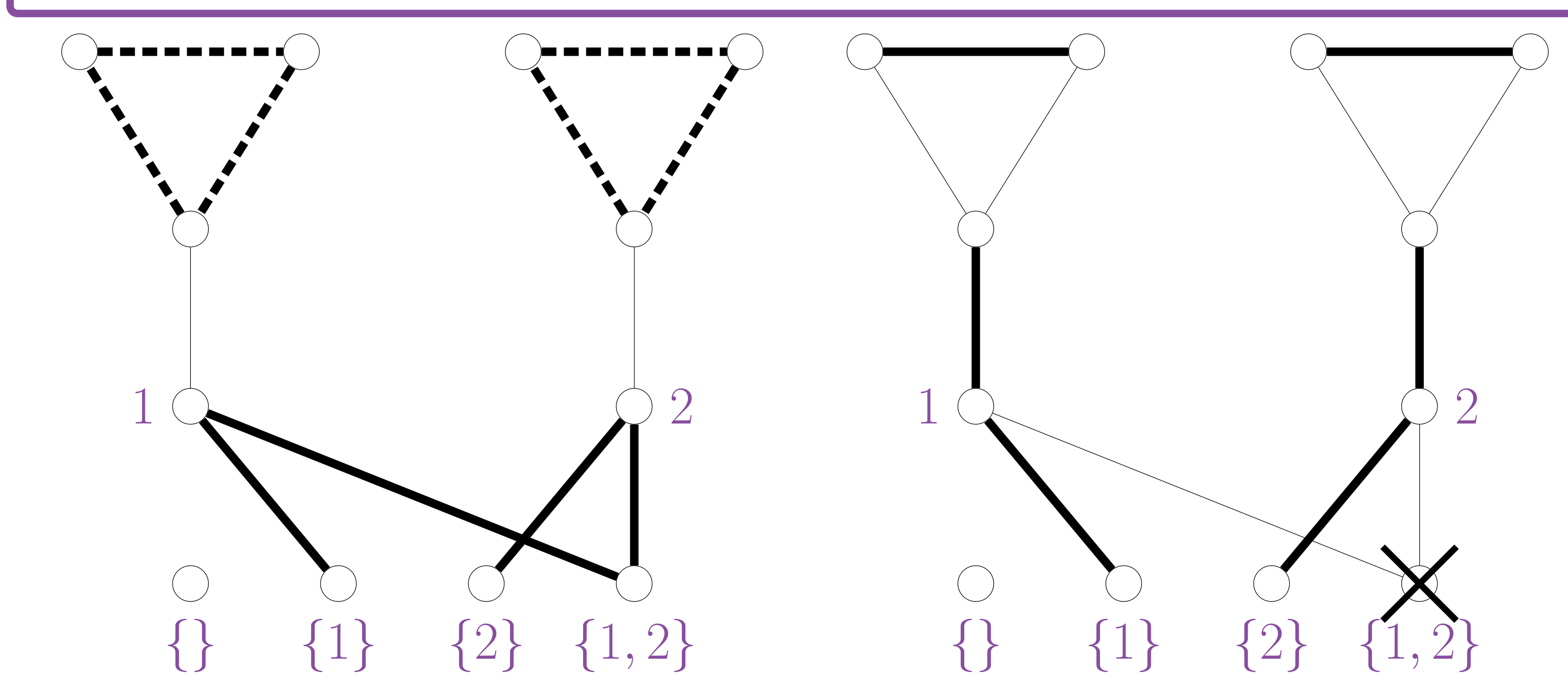
The minimum vertex-stabilizer problem in unit-capacity graphs is polynomial-time solvable for both unit-weight and weighted graphs [1, 5, 7]. In unit-weight graphs the size of the matching is preserved ($\nu(G \setminus S) = \nu(G)$) [1], while in weighted graphs the weight of the matching might not be fully preserved ($\nu(G \setminus S) \geq \frac{2}{3}\nu(G)$) [7].

Vertex-Stabilizer in Capacitated Graphs

The minimum vertex-stabilizer problem in capacitated graphs is NP-complete, even if all edges have unit-weight [4]. We show a stronger inapproximability result: it is as hard to approximate as the set cover problem [3].

Theorem 2

Unless $P = NP$, the vertex-stabilizer problem in capacitated graphs cannot be approximated within a factor $(1 - \varepsilon) \cdot \ln n$ for every $\varepsilon > 0$, even if all edges have unit-weight.



Open Problems

- Computational complexity of deciding when the core of a CMG is nonempty in capacitated graphs.
- Develop an $O(\ln n)$-approximation algorithm for the vertex-stabilizer problem in capacitated graphs, if possible.
- Studying other ways to stabilize capacitated graphs, for example by removing minimum-cardinality edge sets.

References

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