Stabilization of Capacitated Matching Games

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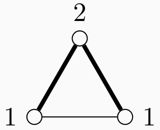
Introduction

Introduction

- We consider a graph $G = (V, E)$ with edge weights w and vertex capacities c.
- We say $M \subseteq E$ is a c-matching if $|M \cap \delta(v)| \leq c_v$ for all $v \in V$.
- $\nu^c(G) = \max\left\{\sum_{e \in M} w_e : M \text{ is a } c\text{-matching in } G\right\}$
- $\nu_f^c(G) = \max\left\{\sum_{e \in E} w_e x_e : \sum_{e \in \delta(v)} x_e \leq c_v \ \forall v \in V, 0 \leq x \leq 1\right\}$
- We call a graph G stable if $\nu^c(G) = \nu_f^c(G)$.

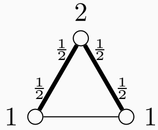


$$\nu^c(G) = 2 = \nu_f^c(G)$$

Network bargaining games

Network bargaining games were first introduced by Kleinberg and Tardos (2008), as a generalization of Nash's 2-player bargaining solution (1950).

- V: players
- E: potential deals with value w
- a player v can enter in c_v deals: a set of deals is a c-matching M
- players decide how to split the value of their deal:
  $z_{uv} + z_{vu} = w_{uv}$ if $uv \in M$ and $z_{uv} = z_{vu} = 0$ otherwise
- stable solution if all players are satisfied



$$\alpha_u(M,z) = \max_{v: uv \in E \setminus M} \left(w_{uv} - \mathbf{1}_{[d_v^M = c_v]} \min_{vw \in M} z_{vw} \right)$$

Theorem (Bateni, Hajiaghayi, Immorlica, Mahini (2010))

There exists a stable solution for the network bargaining game on G if and only if G is stable.

The stabilization problem

The stabilization problem: minimally modify a graph to turn it into a stable one.

Previously studied modifications:

- Edge removal [Biró et al. (2014)] [Bock et al. (2015)] [Koh, Sanità (2020)]
- Vertex removal [Ito et al. (2017)] [Ahmadian et al. (2018)] [Koh, Sanità (2020)]
- Edge and vertex addition [Ito et al. (2017)]
- Increasing edge weights [Chandrasekaran et al. (2019)]

The stabilization problem - vertex removal

A vertex-stabilizer is a set $S \subseteq V$ such that $G \setminus S$ is stable ($\nu^c(G \setminus S) = \nu_f^c(G \setminus S)$).

Vertex-stabilizer problem: given a graph (G, w, c), find a min-cardinality vertex-stabilizer

M-vertex-stabilizer problem: given a graph (G, w, c) and a max-weight c-matching M, find a min-cardinality vertex-stabilizer among those avoiding M

Generalized M-vertex-stabilizer problem: given a graph (G, w, c) and an arbitrary c-matching M, find a min-cardinality vertex-stabilizer S among those for which M is a max-weight c-matching in $G \setminus S$

(G, w, c)	vertex-stabilizer	M-vertex-stabilizer	generalized M-vertex-stabilizer	
$w = 1,\ c = 1$	P [IKK+17][AHS18]	P [AHS18]	tight 2-approx [KS20]	
$w \geq 0,\ c = 1$	P [KS20]	P [KS20]	tight 2-approx [KS20]	
$w \geq 0,\ c \geq 0$	tight $\lvert V \rvert$-approx	P	tight 2-approx	[this work]

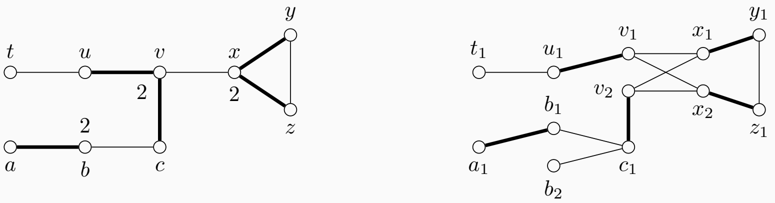
(Generalized) M-vertex-stabilizer

(Generalized) M-vertex-stabilizer

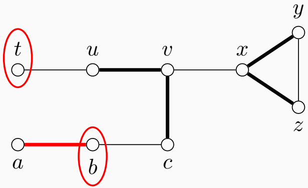
Natural idea for an algorithm:

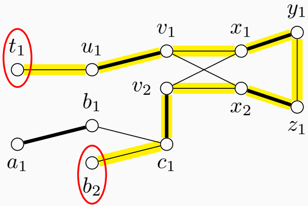
- Transform the given graph and c-matching into an auxiliary unit-capacity instance using a reduction from [Farczadi, Georgiou, Könemann (2013)].
- Apply the algorithm from [Koh, Sanità (2020)] to the unit-capacity auxiliary instance.

This does not work if both algorithms are used as a black-box.



Issues when using the unit-capacity algorithm

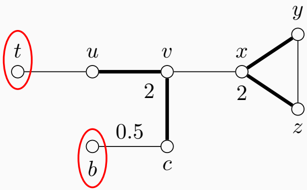


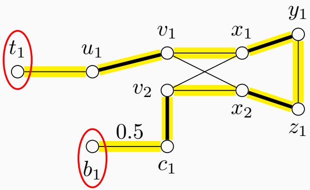


Issue 1: the algorithm suggest to remove a vertex that cannot be removed.

Solution: show that only removing t is enough, or that the graph cannot be stabilized.

Issues when using the unit-capacity algorithm

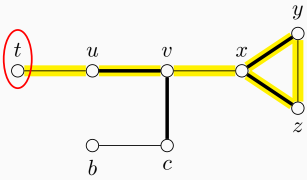


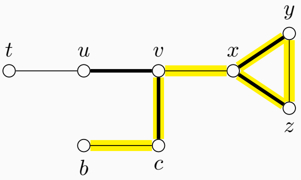


Issue 2: the algorithm suggest to remove two vertices, while one could be enough.
$\rightarrow$ 2-approximation algorithm.

Solution:

- M-vertex-stabilizer problem: use traceback operation to get an exact algorithm.
- Generalized M-vertex-stabilizer problem: satisfied with 2-approximation.





(Generalized) M-vertex-stabilizer

Theorem

*The M-**vertex stabilizer** problem can be solved in polynomial time, and the **generalized M-vertex stabilizer** problem admits an efficient 2-approximation.*

$\rightarrow$ Use the auxiliary construction, and apply the unit-capacity algorithm keeping in mind issues 1 and 2.

Vertex-stabilizer

Theorem

The **vertex-stabilizer** *problem is NP-complete, and no efficient $|V|^{1-\varepsilon}$-approximation exists for any $\varepsilon > 0$, unless $P = NP$. This is true even when all edges have weight one.*

Trivial $|V|$-approximation: remove all vertices.

We will give an approximation preserving reduction from the following problem:

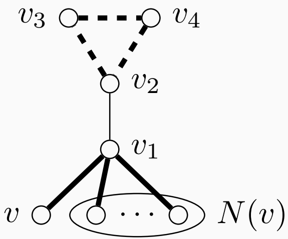
Minimum independent dominating set problem: given a graph $G = (V, E)$, compute a minimum-cardinality subset $S \subseteq V$ that is independent and dominating.

Reduction

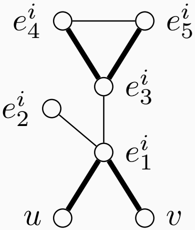
Let $G = (V, E)$.

Let G' be the union of the following gadgets:

- Γ_v for every $v \in V$:



- Γ_e^i for every $e = uv \in E$ and $i \in \{1, \ldots, |V|\}$:



Claim: G has an independent dominating set of size at most k if and only if G' has a vertex-stabilizer of size at most k.

Cooperative matching games

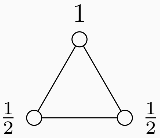
Cooperative matching games

- V: players
- players can distribute a total value of $\nu^c(G)$
- $y \in \mathbb{R}^V_{\geq 0}$: allocation vector

  $\rightarrow$ y_v is the value assigned to player v
- stable solution if every subset of players is satisfied:

$$\text{if } \sum_{v \in S} y_v \geq \nu^c(G[S]) \text{ for all } S \subseteq V$$

- core: set of all stable solutions of total value $\nu^c(G)$



$$\nu^c(G) = 2$$

(i) G is stable

(ii) there exists a stable solution for the network bargaining game on G

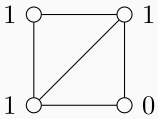
(iii) there exists a solution in the core of the cooperative matching game on G

Theorem (Kleinberg, Tardos (2008), Deng, Ibaraki, Nagamochi (1999))

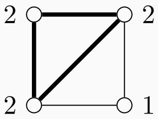
In unit-capacity graphs (i), (ii) and (iii) are equivalent.

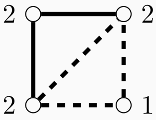
Does this generalize to capacitated graphs?

- Bateni et al. (2010): (i) and (ii) are equivalent, and (i) implies (iii).
- Biró et al. (2016), Gerstbrein, Sanità, Verberk (2022): (iii) does not imply (i).



$(1, 1, 1, 0) \in$ core





$\nu^c(G) = 3 \quad \neq \quad \nu_f^c(G) = 3.5$

Conclusion & future work

Conclusion & future work

- Results for the **(generalized) M-vertex-stabilizer** problem extend to the capacitated case:
  - The **M-vertex-stabilizer** problem is polynomial time solvable.
  - The **generalized M-vertex-stabilizer** problem admits an efficient 2-approximation.
- Results for the **vertex-stabilizer** problem do not extend to the capacitated case:
  - While the problem is polynomial time solvable in the unit-capacity case, there cannot be an efficient $|V|^{1-\varepsilon}$-approximation in the capacitated case.
  - There is a trivial $|V|$-approximation for the **vertex-stabilizer** problem.

Future

- Stabilizing capacitated graphs by reducing the capacity of vertices.
- Stabilizing capacitated graphs by removing edges.
- Stabilizing cooperative matching games in capacitated instances.